# ON IRREDUCIBLE RATIONAL QUINTICS

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Abstract. In this survey paper, we present a classification of irreducible quintics in $\mathbb{Q}[x]$ and provide a procedure for solving quintics in Bring-Jerrard form that are irreducible over $\mathbb{Q}$ by radicals. These results are applied to several original examples.

## 1. Introduction

In 1823, Abel [Abe24] (see also [Abe39, pp. 21–24; Rot98, Theorem 75]) showed that there exists a quintic polynomial over $\mathbb{Q}$ that is not solvable by radicals, a theorem partially proven by Ruffini in 1799 [Ruf99a; Ruf99b] and now known as the *Abel-Ruffini theorem*. Moreover, we know by Galois' theorem [Rot98, Theorem 98] that a polynomial over $\mathbb{Q}$ is solvable by radicals if and only if its Galois group is solvable. Thus, a desire to solve quintics that are irreducible over $\mathbb{Q}$ by radicals leads to the following motivating question:

**Question.** How can we characterize the Galois group of a quintic irreducible over $\mathbb{Q}$?

Considerable work has been done on classifying the Galois group of quintics irreducible over $\mathbb{Q}$, and the task of finding roots of quintics that are solvable by radicals has been one of great interest in the history of mathematics. In 1858, Hermite [Her58] solved irreducible rational quintics in Bring-Jerrard form using Jacobi theta functions. The 1880s then saw a flurry of work surrounding quintics, with Young [You83], McClintock [McC83], Glashan [Gla85], Young [You85] and McClintock [McC85] providing criterion for their solvability by radicals and procedures for finding the roots of solvable quintics. In 1898, Weber presented necessary and sufficient conditions for the solvability of irreducible rational quintics in Bring-Jerrard form by radicals in *Lehrbuch der Algebra* [Web98a, §196] (see also *Traité d'algèbre supérieure* [Web98b, §196]), while half a century later, in 1948, Watson gave a lecture at the University of Cambridge describing a procedure for solving any solvable irreducible rational quintic polynomial by radicals [BSW02]. More recently,

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Dummit [Dum91] provided closed formulas for the roots of any irreducible solvable quintic in $\mathbb{Q}[x]$.

Our classification of the Galois groups of irreducible quintics is in terms of a decic resolvent first announced by Jensen and Yui [JY80] and studied in detail in [JY82], and it makes use of the geometry of $K_5$, the complete graph on 5 vertices. After covering some preliminaries in section 2, we present our classification theorem, Weber's theorem, and several other classification criterion in section 3. We present in section 4 a method by Spearman and Williams [SW94] for solving solvable quintics in Bring-Jerrard form by radicals, and we conclude by applying our classification theorems and the method from section 4 to several examples in section 5.

## 2. Preliminaries

We begin by establishing some preliminary definitions and results used later in the study of irreducible rational quintics.

**Definition 1.** Let $f(x) = x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 \in \mathbb{Q}[x]$ be a rational quintic. Then $f(x)$ is in *Bring-Jerrard form* if $c_4 = c_3 = c_2 = 0$, that's to say, if $f(x) = x^5 + ax + b \in \mathbb{Q}[x]$ for some $a, b \in \mathbb{Q}$.

It was shown by Bring in 1786 (see [Har64] for a translation from the original Latin) and again by Jerrard in 1832 (see [Jer35a; Jer35b]) that any general quintic may be transformed into Bring-Jerrard form; for a modern discussion of their results and of how to transform any general quintic into Bring-Jerrard form, the reader is referred to [AJ03]. Quintic polynomials in Bring-Jerrard form are of interest because they have the following pleasant property, which is immediate from the resultant of their Sylvester matrix:

**Proposition 1.** *If $f(x) = x^5 + ax + b \in \mathbb{Q}[x]$ is a quintic in Bring-Jerrard form, then its discriminant is given by* $\operatorname{Disc}(f) = 4^4a^5 + 5^5b^4$.

Finally, in classifying the Galois groups of irreducible rational quintics, we will make extensive use of [Art07, Lemma 7.10]:

**Lemma 1.** *There is a one-to-one correspondence between the irreducible factors of a separable polynomial $f(x)$ over its base field and the orbits of its Galois group acting on its roots.*

*Proof.* Let $f(x) \in K[x]$, $M/K$ be its splitting field, and $p(x) = (x - \alpha_1)\cdots(x - \alpha_n) \in M[x]$ be one of its irreducible factors in $K[x]$. Then for all $\sigma \in \operatorname{Gal}(f/K)$ and all $1 \le i \le n$, $p(\alpha_i) = 0$ implies $p(\sigma(\alpha_i)) = 0$. Thus $\sigma(\alpha_i)$ is one of $\alpha_1, \dots, \alpha_n$, and since $p(x)$ is the minimal polynomial of $\alpha_i$ over $K$, this implies that $\alpha_i$ has exactly $n$ distinct images under $\operatorname{Gal}(f/K)$, namely, $\alpha_1, \dots, \alpha_n$. Thus we see that $p(x)$ uniquely determines the orbits of $\alpha_1, \dots, \alpha_n$. $\square$

**Corollary 1** (of proof)**.** *The size of the orbit of a root is the the degree of its associated irreducible factor.*

## 3. Classifying Galois Groups of Irreducible Quintics

To help classify the Galois group of irreducible rational quintics, we introduce the following decic resolvent by Jensen and Yui [JY80; JY82] whose value will soon become apparent:

**Definition 2.** Let $\alpha_1, \dots, \alpha_5$ be the roots of a monic irreducible quintic $f(x) \in \mathbb{Q}[x]$ and let

$$P_{10}(x) = \prod_{1 \le i < j \le 5} (x - (\alpha_i + \alpha_j)). \tag{1}$$

Then we call $P_{10}(x)$ the *decic resolvent* of $f(x)$.

The following theorem shows that $P_{10}(x)$ and $f(x)$ have the same Galois group. This resolvent is useful in classifying the Galois groups of irreducible rational quintics in that we can easily determine the Galois group of $P_{10}(x)$ by observing its action on the complete graph on 5 vertices, $K_5$, where we identify the root $(\alpha_i + \alpha_j)$ of $P_{10}(x)$ with the edge $\overline{\alpha_i \alpha_j}$ of $K_5$ (see Figure 1).

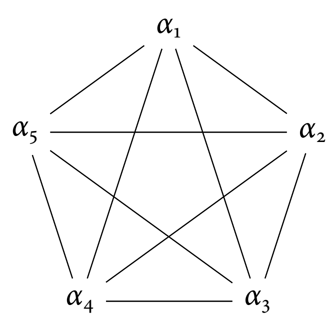


Figure 1. The complete graph on 5 vertices $K_5$

**Theorem 1.** *The resolvent $P_{10}(x)$ is a separable polynomial over $\mathbb{Q}$ whose Galois group is isomorphic to that of the irreducible quintic that induced it.*

*Proof.* There are three things to show:

(1) $P_{10}(x) \in \mathbb{Q}[x]$;
(2) $P_{10}(x)$ is separable; and
(3) $P_{10}(x)$ and $f(x)$ have the same splitting field.

First, let $G = \mathrm{Gal}(P_{10}/\mathbb{Q})$. Then for any $\sigma \in G$, we clearly have $\sigma(P_{10}(x)) = \prod_{1 \le i < j \le 5}(x - \sigma(\alpha_i + \alpha_j)) = P_{10}(x)$ since $P_{10}(x)$ already contains all possible sums of pairs of roots of $f(x)$ as roots and $\sigma$ is an automorphism. Thus, $\sigma$

fixes $P_{10}(x)$'s coefficients, and so they must lie in $\mathbb{Q}$. So $P_{10}(x) \in \mathbb{Q}[x]$ as desired.

We now show that $P_{10}(x)$ is separable; it suffices to show that the sums of pairs of roots $\alpha_i + \alpha_j$ for $i \neq j$ are pairwise distinct. We present a specialization of the general proof by G. A. Elliot in [JY82, Lemma II.1.1]. Let $f/\mathbb{Q}$ be the splitting field of $f(x)$ over $\mathbb{Q}$, let $V$ be the subspace $V$ of $f/\mathbb{Q}$ given by

$$V = \mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2 + \cdots + \mathbb{Q}\alpha_5,$$

and consider the $\mathbb{C}$-vector space $\overline{V} = \mathbb{C} \otimes_{\mathbb{Q}} V$ over $\mathbb{Q}$. Then the automorphism $\sigma = (12345) \in \mathrm{Gal}(f/\mathbb{Q})$ of order 5 acting on the $\alpha_i$ in the obvious manner is an automorphism of $V$, and it can be lifted to the automorphism $\bar{\sigma} = 1_{\mathbb{C}} \otimes \sigma$ of $\overline{V}$ which also has order 5. Since $\bar{\sigma}^5 = 1$, $\bar{\sigma}$ is diagonalizable, and its eigenvalues are 5th roots of unity. Let $v_1, \ldots, v_t$ be the eigenbasis associated with $\bar{\sigma}$ in $\overline{V}$, then since each $v_i$ is an eigenvector of $\bar{\sigma}$, we have $\bar{\sigma} v_i = \epsilon_i v_i$ for some corresponding 5th root of unity $\epsilon_i$.

We identify $V$ and its copy in $\overline{V}$, i.e., we identify $\sum_{k=1}^{5} c_i \alpha_i$ in $V$ and $1_{\mathbb{C}} \otimes \sum_{k=1}^{5} c_i \alpha_i$ in $\overline{V}$. Now consider the linear expansion of $\alpha_1$ in terms of the eigenbasis of $\bar{\sigma}$: $\alpha_1 = \sum_{i=1}^{t} c_i v_i$ for some $c_i \in \mathbb{C}$. Then

$$\bar{\sigma}\alpha_1 = \sum_{i=1}^{t} c_i \bar{\sigma} v_i = \sum_{i=1}^{t} c_i \epsilon_i v_i, \tag{2}$$

and there must exist a $\lambda$ such that $c_\lambda \neq 0$ and $\epsilon_\lambda \neq 1$ in (2), because otherwise $\bar{\sigma}\alpha_1 = \alpha_1$. Moreover, $\alpha_k = \bar{\sigma}^k \alpha_1 = \sum_{i=1}^{t} c_i \epsilon_i^k v_i$.

Now, assume that two distinct pairs of $\alpha$s have the same sum, say $\alpha_i + \alpha_j = \alpha_k + \alpha_l$. Then comparing the $\lambda$th coordinates, we have that

$$c_\lambda \epsilon_\lambda^i + c_\lambda \epsilon_\lambda^j = c_\lambda \epsilon_\lambda^k + c_\lambda \epsilon_\lambda^l,$$
$$\epsilon_\lambda^i + \epsilon_\lambda^j = \epsilon_\lambda^k + \epsilon_\lambda^l.$$

Hence we have the identity:

$$\epsilon_\lambda^i + \epsilon_\lambda^j - \epsilon_\lambda^k - \epsilon_\lambda^l = 0.$$

This implies that $\epsilon_\lambda$ is a root of the non-zero polynomial $x^i + x^j - x^k - x^l \in \mathbb{Q}[x]$, a contradiction since the minimal polynomial of $\epsilon_\lambda$ over $\mathbb{Q}$ is $x^4 + x^3 + x^2 + x + 1$. Thus, all sums of pairs of roots $\alpha_i + \alpha_j$, $i \neq j$, are pairwise distinct, so $P_{10}(x)$ is separable.

Finally, we show that $P_{10}(x)$ and $f(x)$ have the same splitting field. Denote the respective splitting fields as $P_{10}/\mathbb{Q}$ and $f/\mathbb{Q}$, then clearly $P_{10}/\mathbb{Q} \subseteq f/\mathbb{Q}$. To see that $f/\mathbb{Q} \subseteq P_{10}/\mathbb{Q}$, it suffices to observe that $\alpha_i \in P_{10}/\mathbb{Q}$ for all $i$:

$$\alpha_i = \frac{1}{2}\left(\left(\alpha_i + \alpha_j\right) + \left(\alpha_i + \alpha_k\right) - \left(\alpha_j + \alpha_k\right)\right) \in P_{10}/\mathbb{Q}.$$

Thus $P_{10}/\mathbb{Q} = f/\mathbb{Q}$. We conclude that $\mathrm{Gal}(P_{10}/\mathbb{Q}) \cong \mathrm{Gal}(f/\mathbb{Q})$. $\square$

By Lemma 1, $\mathrm{Gal}(f/\mathbb{Q})$ is transitive whenever $f(x)$ is irreducible in $\mathbb{Q}[x]$, and so it must be one of the subgroups shown in Figure 2. Since the only solvable subgroups of $S_5$ are those of $F_{20}$, it is easy to see that an irreducible quintic over $\mathbb{Q}$ is solvable by radicals if and only if its Galois group is a subgroup of $F_{20}$. We can now proceed to classify the Galois groups of irreducible quintics.

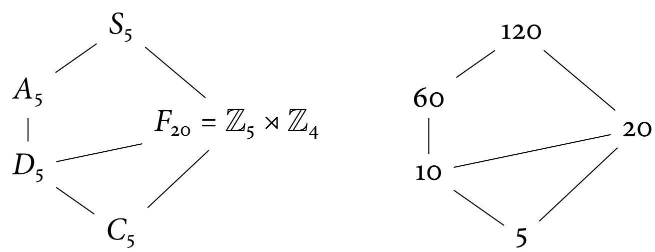


FIGURE 2. Transitive subgroups of $S_5$ and their respective orders

**Theorem 2.** *Let $f(x) \in \mathbb{Q}[x]$ be a quintic irreducible over $\mathbb{Q}$. Assume* $\mathrm{Disc}(f) \in (\mathbb{Q}^*)^2$, *then:*

(1) *$P_{10}(x)$ is irreducible over $\mathbb{Q}$ if and only if* $\mathrm{Gal}(f/\mathbb{Q}) \cong A_5$.

*Otherwise, $P_{10}(x)$ is the product of two quintics irreducible over $\mathbb{Q}$ and*

(2) *$f(x)$ has a complex root if* $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$;
(3) *$f(x)$ has 5 real roots if* $\mathrm{Gal}(f/\mathbb{Q}) \cong \mathbb{Z}_5$.

*Now assume that* $\mathrm{Disc}(f) \notin (\mathbb{Q}^*)^2$, *then:*

(4) *$P_{10}(x)$ is irreducible over $\mathbb{Q}$ if and only if* $\mathrm{Gal}(f/\mathbb{Q}) \cong S_5$;
(5) *Otherwise, $P_{10}(x)$ is the product of two quintics irreducible over $\mathbb{Q}$ and* $\mathrm{Gal}(f/\mathbb{Q}) \cong F_{20}$.

*Proof.* First assume that $\mathrm{Disc}(f) \in (\mathbb{Q}^*)^2$, then $\mathrm{Gal}(f/\mathbb{Q})$ is isomorphic to a subgroup of $A_5$. We begin by determining the number of orbits of $A_5$, $D_5$ and $\mathbb{Z}_5$ acting on the roots of $P_{10}(x)$ by observing their action on $K_5$.

First, we consider $A_5$, which can be realized as $A_5 \cong \langle (12345), (123) \rangle < S_5$. As exhibited in Figure 3, the permutation $(123)$ fixes only the edge $\overline{\alpha_4\alpha_5}$ and permutes all others, while $(12345)$ permutes all edges of the pentagon. Thus, $A_5$ has one orbit when acting on the roots of $P_{10}(x)$, and so by Lemma 1, $P_{10}(x)$ is irreducible if $\mathrm{Gal}(f/\mathbb{Q}) \cong A_5$.

We now consider $D_5$, which can be realized as $D_5 \cong \langle (12345), (25)(34) \rangle < S_5$. We recall the fact that $D_5$ is the reflection group of the regular pentagon and is distance preserving. Thus, from a geometric point of view, it is clear that $D_5$ has at least two orbits since it cannot send edges from the pentagram to the pentagon nor vice-versa. Geometrically, it is clear that the pentagon's

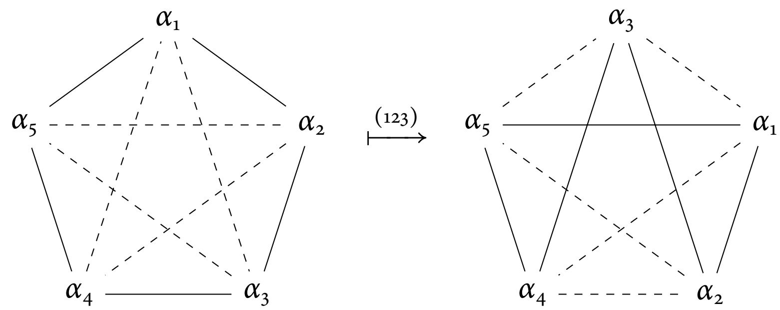


FIGURE 3. The action of $(123) \in A_5$ acting on $K_5$

edges form an orbit under rotation, and similarly for the pentagram. Thus, $D_5$ has two orbits of order 5 when acting on the roots of $P_{10}(x)$, and we conclude by Lemma 1 that $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$ only if $P_{10}(x)$ factors into two quintics irreducible over $\mathbb{Q}$.

Finally, we consider $\mathbb{Z}_5$, which can be realized as $\mathbb{Z}_5 \cong \langle (12345) \rangle < S_5$. It is easy to see that $\mathbb{Z}_5$ has two orbits of order 5 since it only acts on the pentagon and pentagram by rotation. It is also easy to see that $f(x)$ have all real roots is a necessary condition for $\mathrm{Gal}(f/\mathbb{Q}) \cong \mathbb{Z}_5$, since complex roots imply $\mathrm{Gal}(f/\mathbb{Q})$ contains an involution and $2 \nmid |\mathbb{Z}_5|$.

Combining these analyses, we see that $A_5$ is the only group with one orbit when acting on the roots of $P_{10}(x)$, so $\mathrm{Gal}(f/\mathbb{Q}) \cong A_5$ if and only if $P_{10}(x)$ is irreducible over $\mathbb{Q}$. Both $D_5$ and $\mathbb{Z}_5$ have two orbits, and we presented necessary conditions for each in terms of the roots of $f(x)$.

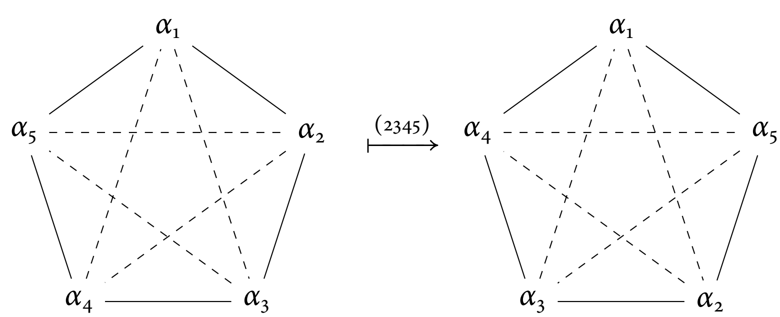


FIGURE 4. The action of $(2345) \in F_{20}$ acting on $K_5$

Now assume that $\mathrm{Disc}(f) \notin (\mathbb{Q}^*)^2$. Since $S_5$ is intransitive, it has one orbit when acting on the roots of $P_{10}(x)$. However, as shown in Figure 4, $F_{20} \cong \langle (12345), (2345) \rangle < S_5$ has two orbits, each of order five: the pentagon and the pentagram. Thus, conclusions (4) and (5) are immediate by Lemma 1 and Corollary 1. $\square$

Although in general, computing the resolvent $P_{10}(x)$ of an irreducible quintic $f(x) \in \mathbb{Q}[x]$ is difficult, Jensen and Yui [JY80; JY82] assert without proof that the resolvent $P_{10}(x)$ of a quintic $f(x) = x^5 + ax + b \in \mathbb{Q}[x]$ irreducible over $\mathbb{Q}$ can be expressed as

$$P_{10}(x) = x^{10} - 3ax^6 - 11bx^5 - 4a^2x^2 + 4abx - b^2 \in \mathbb{Z}[a,b][x]. \tag{3}$$

We can further simplify the task of classifying such quintics via Corollary 2:

**Lemma 2** ([JY82, Lemma II.2.4])**.** *Let $f(x) = x^p + ax + b \in \mathbb{Q}[x]$ with $p \geq 3$ prime, and assume that $f(x)$ is irreducible over $\mathbb{Q}$. Then $f(x)$ has at most three real roots and only one if $a > 0$.*

*Proof.* Since $f(x)$ has odd degree, it must have at least one real root. By analysis, since $df/dx = px^{p-1} + a$ has at most two real roots and none if $a > 0$, the assertion follows. □

The following corollary of Lemma 2 is a correction of [JY82, Corollary II.2.5].

**Corollary 2.** *Let $f(x) = x^p + ax + b \in \mathbb{Q}[x]$. If $p > 3$ is prime, or if $p = 3$ and $a = 1$, then $\mathrm{Gal}(f/\mathbb{Q}) \ncong \mathbb{Z}_p$.*

*Proof.* By the lemma, $f(x)$ has at least one complex conjugate pair of roots, and so $\mathrm{Gal}(f/\mathbb{Q})$ contains an involution. Then $2 \mid |\mathrm{Gal}(f/\mathbb{Q})|$, but $2 \nmid |\mathbb{Z}_p|$, so $\mathrm{Gal}(f/\mathbb{Q}) \ncong \mathbb{Z}_p$. □

In fact, even though the classification presented above for generic irreducible rational quintics is incomplete, Weber's theorem [Web98a, §189; Web98b, §189], as given by Jensen, Ledet and Yui [JLY02, Theorem 2.3.4], completes the classification for irreducible rational quintics in Bring-Jerrard form:

**Theorem 3.** *Let $f(x) = x^5 + ax + b \in \mathbb{Q}[x]$ be irreducible. If $a = 0$, then $\mathrm{Gal}(f/\mathbb{Q}) \cong F_{20}$. Otherwise, $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$ (resp. $F_{20}$) if and only if*

*(i)* $\mathrm{Disc}(f) \in (\mathbb{Q}^*)^2$ *(resp. $\notin (\mathbb{Q}^*)^2$)*
*(ii) $a$ and $b$ have the form*

$$a = \frac{5^5 \lambda \mu^4}{(\lambda - 1)^4(\lambda^2 - 6\lambda + 25)}, \quad b = a\mu$$

*for some $\lambda, \mu \in \mathbb{Q}$ with $\lambda \neq 1$ and $\mu \neq 0$.*

Weber's theorem is logically equivalent to one by Spearman and Williams [SW94, p. 987ff.], presented below (Theorem 4), which can be used to obtain the roots of an irreducible rational quintic in Bring-Jerrard form.

For a complete classification of general irreducible rational quintics, the reader is referred to [Dum91].

## 4. Solving Solvable Quintics by Radicals

The following theorem is by Spearman and Williams [SW94], and its sufficiency condition was known in 1885 by Glashan [Gla85]. We present an expanded version of its proof since it provides a constructive approach to finding the constants $\epsilon$, $c$, and $e$ (see (10) and Example 2), and thus to finding the roots of an irreducible quintic in $\mathbb{Q}[x]$ in Bring-Jerrard form.

**Theorem 4.** *Let $a, b \in \mathbb{Q}$ such that the quintic trinomial $x^5+ax+b$ is irreducible over $\mathbb{Q}$. Then the equation $x^5 + ax + b = 0$ is solvable by radicals if and only if there exist rational numbers $\epsilon = \pm 1$, $c \geq 0$ and $e \neq 0$ such that*

$$a = \frac{5e^4\left(3 - 4\epsilon c\right)}{c^2+1}, \quad b = \frac{-4e^5\left(11\epsilon + 2c\right)}{c^2+1}, \tag{4}$$

*in which case $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$ if and only if $5D \in (\mathbb{Q}^*)^2$, where $D = c^2 + 1$, otherwise, $\mathrm{Gal}(f/\mathbb{Q}) \cong F_{20}$. If the equation is solvable by radicals, then the roots are*

$$x_j = e\left(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4\right) \tag{5}$$

*for $j = 0, 1, 2, 3, 4$, where $\omega = \exp\left(2\pi i/5\right)$ and*

$$\begin{cases} u_1 &= \left(\frac{v_1^2 v_3}{D^2}\right)^{(1/5)} \quad u_2 = \left(\frac{v_3^2 v_4}{D^2}\right)^{(1/5)} \\ u_3 &= \left(\frac{v_2^2 v_1}{D^2}\right)^{(1/5)} \quad u_4 = \left(\frac{v_4^2 v_2}{D^2}\right)^{(1/5)} \end{cases}, \tag{6}$$

$$\begin{cases} v_1 &= \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}} \quad v_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}} \\ v_3 &= -\sqrt{D} + \sqrt{D + \epsilon\sqrt{D}} \quad v_4 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}} \end{cases}. \tag{7}$$

*Proof.* We first assume that $f(x) = x^5 + ax + b$ is solvable by radicals. Then by [Dum91, Theorem 1], this implies that the resolvent sextic

$$\begin{aligned} f_{20}(x) &= x^6 + 8ax^5 + 40a^2x^4 + 160a^3x^3 + 400a^4x^2 + \\ &\quad + \left(512a^5 - 3125b^4\right)x + \left(256a^6 - 9375ab^4\right) \\ &= \left(x + 2a\right)^4\left(x^2 + 16a^2\right) - 5^5b^4\left(x + 3a\right) \in \mathbb{Q}[x] \end{aligned} \tag{8}$$

of $f(x)$ has a rational root $r$. Then $f_{20}(r) = 0$, and rearranging terms gives us

$$\frac{5^5b^4\left(r + 3a\right)}{\left(r + 2a\right)^4\left(r^2 + 16a^2\right)} = 1, \tag{9}$$

and when $a \neq 0$, we have $r \neq -2a, -3a$. We let the rational numbers $c \geq 0$, $e \neq 0$, $\epsilon = \pm 1$ be given by

$$\epsilon c = \frac{3r - 16a}{4(r + 3a)}, \quad e = \frac{-5b\epsilon}{2(r + 2a)}. \tag{10}$$

Then by simple algebraic manipulation, we get that

$$c^2 + 1 = \frac{25(r^2 + 16a^2)}{16(r + 3a)^2}, \tag{11}$$

$$3 - 4\epsilon c = \frac{25a}{r + 3a}, \tag{12}$$

and

$$11\epsilon + 2c = \frac{25(r + 2a)\epsilon}{2(r + 3a)}. \tag{13}$$

Then by (11), (12), and (9),

$$\frac{5e^4(3 - 4\epsilon c)}{c^2 + 1} = a\frac{5^5 b^4 (r + 3a)}{(r + 2a)^4 (r^2 + 16a^2)} = a$$

and by (11), (13), and (9),

$$\frac{-4e^5(11\epsilon + 2c)}{c^2 + 1} = b\frac{5^5 b^4 (r + 3a)}{(r + 2a)^4 (r^2 + 16a^2)} = b.$$

Thus, the parametrization (4) of $a$ and $b$ holds whenever $f(x)$ is solvable.

We now show that the rational quintic

$$\tilde{f}(x) = x^5 + \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1}x + \frac{-4e^5(11\epsilon + 2c)}{c^2 + 1}$$

is solvable by radicals with roots given by (5) when $e = 1$; a transformation $x \mapsto ex$ provides the result for general $e$. Equations (7) imply that

$$\begin{cases} v_1 + v_4 &= 2\sqrt{D} \quad v_2 + v_2 &= -2\sqrt{D} \\ v_1 v_4 &= \epsilon\sqrt{D} \quad v_2 v_3 &= -\epsilon\sqrt{D} \end{cases} \tag{14}$$

and so

$$\begin{cases} v_1 + v_2 + v_3 + v_4 &= 0 \\ v_1 v_4 + v_2 v_3 &= 0 \end{cases}. \tag{15}$$

Taking fifth powers in (6), we get

$$u_1^5 = \frac{v_1^2 v_3}{D^2}, \qquad u_2^5 = \frac{v_3^2 v_4}{D^2}, \qquad u_3^5 = \frac{v_2^2 v_1}{D^2}, \qquad u_4^5 = \frac{v_4^2 v_2}{D^2}. \tag{16}$$

By (14), (16), and the fact that $\epsilon^3 = \epsilon$, we get

$$\begin{aligned}(u_1u_4)^5 &= \frac{v_1^2 v_3}{D^2}\frac{v_4^2 v_2}{D^2}\\ &= D^{-4}(v_1v_4)^2(v_2v_3)\\ &= D^{-4}\left(\epsilon\sqrt{D}\right)^2\left(-\epsilon\sqrt{D}\right)\\ &= \frac{-\epsilon^3}{D^{5/2}}\\ &= \frac{-\epsilon}{D^{5/2}}\end{aligned}$$

and so since the real fifth roots of $\epsilon$ and $-1$ are $\epsilon$ and $-1$ respectively, we get

$$u_1u_4 = -\frac{\epsilon}{\sqrt{D}}.$$

Similarly easy computations using (14) and (16) give us that

$$\text{(17)}\qquad \begin{cases} & & u_1u_4 &= -\dfrac{\epsilon}{\sqrt{D}} & u_2u_3 &= \dfrac{\epsilon}{\sqrt{D}} & & \\ u_1^2u_3 &= \dfrac{v_1}{D} & u_2^2u_1 &= \dfrac{v_3}{D} & u_3^2u_4 &= \dfrac{v_2}{D} & u_4^2u_2 &= \dfrac{v_4}{D}\end{cases}$$

and

$$\text{(18)}\qquad \begin{cases} u_1^3u_2 &= \dfrac{\epsilon v_1 v_3}{D\sqrt{D}} & u_3^3u_1 &= -\dfrac{\epsilon v_1 v_2}{D\sqrt{D}}\\ u_4^3u_3 &= \dfrac{\epsilon v_2 v_4}{D\sqrt{D}} & u_2^3u_4 &= -\dfrac{\epsilon v_3 v_4}{D\sqrt{D}}\end{cases}.$$

Now consider the quintic polynomial whose roots are given by (5), that's to say the quintic

$$\text{(19)}\qquad F(x) = \prod_{j=0}^{4}\left(x - \left(\omega^j u_1 + \omega^{2j}u_2 + \omega^{3j}u_3 + \omega^{4j}u_4\right)\right),$$

where $u_1, u_2, u_3, u_4$ are nonzero real numbers and $\omega$ is a complex fifth root of unity. We show that $\tilde{f}(x) = F(x)$.

By [SW94, p. 986f.], symbolically evaluating $F(x)$ at its roots $R_j = \omega^j u_1 + \omega^{2j}u_2 + \omega^{3j}u_3 + \omega^{4j}u_4$ gives us the following identity valid for $j = 0, 1, 2, 3, 4$:

$$\text{(20)}\qquad R_j^5 - 5UR_j^3 - 5VR_j^2 + 5WR_j + 5(X - Y) - Z = 0$$

where

$$\begin{aligned}
U &= u_1u_4 + u_2u_3,\\
V &= u_1^2u_3 + u_2^2u_1 + u_3^2u_4 + u_4^2u_2,\\
W &= u_1^2u_4^2 + u_2^2u_3^2 - u_1^2u_2 - u_2^3u_4 - u_3^3u_1 - u_4^3u_3 - u_1u_2u_3u_4,\\
X &= u_1^3u_3u_4 + u_2^3u_1u_3 + u_3^3u_2u_4 + u_4^3u_1u_2,\\
Y &= u_1u_3^2u_4^2 + u_2u_1^2u_3^2 + u_3u_2^2u_4^2 + u_4u_1^2u_2^2,\\
Z &= u_1^5 + u_2^5 + u_3^5 + u_4^5.
\end{aligned}$$

However, by (17), $U = 0$, and by (17) and (15), $V = \frac{1}{D}\left(v_1 + v_2 + v_3 + v_4\right) = 0$. Using (7), (16), (17), and (18), it can be shown that

$$\begin{aligned}
5W &= 5\left(u_1^3u_3u_4 + u_2^3u_1u_3 + u_3^3u_2u_4 + u_4^3u_1u_2\right)\\
&= \frac{5\left(3 - 4\epsilon\sqrt{D-1}\right)}{D}\\
&= \frac{5\left(3 - 4\epsilon c\right)}{c^2 + 1}
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
5(X - Y) - Z &= 5\Big(\left(u_1^3u_3u_4 + u_2^3u_1u_3 + u_3^3u_2u_4 + u_4^3u_1u_2\right) -\\
&\qquad \left(u_1u_3^2u_4^2 + u_2u_1^2u_3^2 + u_3u_2^2u_4^2 + u_4u_1^2u_2^2\right)\Big) -\\
&\qquad \left(u_1^5 + u_2^5 + u_3^5 + u_4^5\right)\\
&= -\frac{\left(44\epsilon + 8\sqrt{D-1}\right)}{D}\\
&= -\frac{4\left(11\epsilon + 2c\right)}{c^2 + 1}.
\end{aligned} \tag{22}$$

Thus, by (15), (17), (21), and (22), we have that the identity (20) can be expressed as

$$R_j^5 + \frac{5\left(3 - 4\epsilon c\right)}{c^2 + 1}R_j + \frac{-4\left(11\epsilon + 2c\right)}{c^2 + 1} = 0 \tag{23}$$

for $j = 0, 1, 2, 3, 4$. It follows that $\tilde{f}(x) = F(x)$, and thus that the roots of our original quintic $x^5 + ax + b$ are given by (5).

Conversely, assume that $\tilde{f}(x)$ in (4) is irreducible over $\mathbb{Q}$. Then by the discussion above, it is solvable by radicals, and so its Galois group is solvable. Thus, its Galois group is isomorphic to the Frobenius group $F_{20}$, the dihedral group of the pentagon $D_5$, or the cyclic group of order 5, $\mathbb{Z}_5$. However, since $\tilde{f}(x)$ is in Bring-Jerrard form, by Corollary 2, its Galois group cannot be $\mathbb{Z}_5$. By Proposition 1, the discriminant of a quintic of the form $x^5 + ax + b$ is

$4^4a^5 + 5^5b^4$, and so the discriminant of $\tilde{f}(x)$ is

$$\mathrm{Disc}(\tilde{f}) = \frac{4^4 5^5 e^{20}}{D^5}\left(4\epsilon c^3 - 84c^2 - 37\epsilon c - 122\right)^2 .$$

Since the Galois group of $\tilde{f}(x)$ is a subgroup of $A_5$ if and only $\mathrm{Disc}(\tilde{f})$ is a perfect square in $\mathbb{Q}$, it follows that the Galois group of $\tilde{f}(x)$ is isomorphic to $D_5$ if and only if $\mathrm{Disc}(\tilde{f})$ is a perfect square in $\mathbb{Q}$. But this occurs if and only if $5D$ is a perfect square in $\mathbb{Q}$, and so the theorem follows. □

## 5. Examples

We now consider several examples. In Examples 1 and 2, we apply Theorem 2 to determine the Galois group of irreducible rational quintics, and in Example 2, we further apply Theorem 4 to solve the quintic by radicals. In Example 3, we show that we cannot rely on numerical methods to compute the decic resolvent $P_{10}(x)$.

**Example 1.** Consider the polynomial $f(x) = x^5 + 121x + 55 \in \mathbb{Q}[x]$, which is irreducible over $\mathbb{Q}$ by Eisenstein's criterion with $p = 11$. Then by Proposition 1, $\mathrm{Disc}(f) = 4^4 121^5 + 5^5 55^4 = 3 \cdot 11^4 \cdot 109 \cdot 1392883 \notin (\mathbb{Q}^*)^2$, and so $\mathrm{Gal}(f/\mathbb{Q})$ must be isomorphic to one of $S_5$ or $F_{20}$. By (3),

$$P_{10}(x) = x^{10} - 363x^6 - 605x^5 - 58564x^2 + 26620x - 3025.$$

Using a computer algebra system, we see that $P_{10}(x)$ is irreducible over $\mathbb{Q}$, and so we conclude by Theorem 2 that $\mathrm{Gal}(f/\mathbb{Q}) \cong S_5$ and that $f(x)$ is not solvable by radicals.

**Example 2.** Consider the polynomial $f(x) = x^5 - 3125x - 37500 \in \mathbb{Q}[x]$ which can be shown to be irreducible over $\mathbb{Q}$ using a computer algebra system. Then by Proposition 1, we have $\mathrm{Disc}(f) = 4^4(-3125)^5 + 5^5(-37500)^4 = (2^6 5^{13})^2 \in (\mathbb{Q}^*)^2$, and so $\mathrm{Gal}(f/\mathbb{Q})$ must be one of $A_5$ or $D_5$ since $\mathrm{Gal}(f/\mathbb{Q})$ is isomorphic to a subgroup of $A_5$ and Corollary 2 prohibits $\mathbb{Z}_5$. By (3),

$$\begin{aligned} P_{10}(x) &= x^{10} + 9375x^6 + 412500x^5 - 39062500x^2 + \\ &\quad + 468750000x - 1406250000 \\ &= \left(x^5 - 125x^3 + 1250x^2 + 18750x + 112500\right) \\ &\quad \times \left(x^5 + 125x^3 - 1250x^2 + 6250x - 12500\right) \end{aligned}$$

and so we conclude by Theorem 2 that $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$.

Since $\mathrm{Gal}(f/\mathbb{Q}) \cong D_5$, $f(x)$ is solvable; we find its roots using Theorem 4. We begin by finding the rational root of the resolvent sextic $f_{20}(x)$ of $f(x)$

given by (8),

$$\begin{aligned}f_{20}(x) &= (x-6250)^4(x^2+15625\cdot 4)-\\ &\quad 61798095703125\cdot 10^8(x-9375)\\ &= x^6-25\cdot 10^3x^5+390625\cdot 10^3x^4-48828125\cdot 10^5x^3+\\ &\quad +3814697265625\cdot 10^4x^2-63323974609375\cdot 10^8x+\\ &\quad +5817413330078125\cdot 10^{10}.\end{aligned}$$

Applying the rational roots test, we find that 25000 is a rational root of $f_{20}(x)$. We now solve for $\epsilon$, $c$ and $e$ using (10):

$$\begin{aligned}\epsilon c &= \frac{3r-16a}{4(r+3a)}\\ &= \frac{3\cdot 25000-16(-3125)}{4(25000+3(-3125))}\\ &= 2\end{aligned}$$

and

$$\begin{aligned}e &= \frac{-5b\epsilon}{2(r+2a)}\\ &= \frac{-5(-37500)\epsilon}{2(25000+2(-3125))}\\ &= -5\epsilon.\end{aligned}$$

Since $c \geq 0$ and $\epsilon = \pm 1$, we conclude that $\epsilon = 1$, $c = 2$, and $e = -5$. Then the roots of $f(x)$ are given by

$$x_j = -5\left(\omega^j u_1 + \omega^{2j}u_2 + \omega^{3j}u_3 + \omega^{4j}u_4\right)$$

for $j = 0, 1, 2, 3, 4$, where $\omega = \exp(2\pi i/5)$ and

$$\begin{aligned}u_1 &= \left(\frac{\left(\sqrt{5}+\sqrt{5-\sqrt{5}}\right)^2\left(-\sqrt{5}+\sqrt{5+\sqrt{5}}\right)}{25}\right)^{(1/5)}\\ &= \left(\frac{25-10\sqrt{5}+3\sqrt{5\left(25-11\sqrt{5}\right)}}{25}\right)^{(1/5)},\\ u_2 &= \left(\frac{\left(-\sqrt{5}+\sqrt{5+\sqrt{5}}\right)^2\left(\sqrt{5}-\sqrt{5-\sqrt{5}}\right)}{25}\right)^{(1/5)}\\ &= \left(\frac{25+10\sqrt{5}-3\sqrt{5\left(25+11\sqrt{5}\right)}}{25}\right)^{(1/5)},\end{aligned}$$

$$u_3 = \left(\frac{\left(-\sqrt{5}-\sqrt{5+\sqrt{5}}\right)^2\left(\sqrt{5}+\sqrt{5-\sqrt{5}}\right)}{25}\right)^{(1/5)}$$
$$= \left(\frac{25+10\sqrt{5}+3\sqrt{5\left(25+11\sqrt{5}\right)}}{25}\right)^{(1/5)},$$

and

$$u_4 = \left(\frac{\left(\sqrt{5}-\sqrt{5-\sqrt{5}}\right)^2\left(-\sqrt{5}-\sqrt{5+\sqrt{5}}\right)}{25}\right)^{(1/5)}$$
$$= \left(\frac{25-10\sqrt{5}-3\sqrt{5\left(25-11\sqrt{5}\right)}}{25}\right)^{(1/5)}.$$

**Example 3.** Consider the polynomial $f(x) = x^5-2x^4-78x^3+159x^2-80x+1 \in \mathbb{Q}[x]$, which is an instance of the generic family

$$f(x;s,t) = x^5 + (t-3)x^4 + (s-t+3)x^3 + \\ + (t^2-t-2s-1)x^2 + sx + t \in \mathbb{Q}(s,t)[x]$$

for $D_5$ given by [JLY02, Theorem 2.3.5] instantiated with $s = -80$ and $t = 1$. Then $f(x) \equiv x^5 + x^2 + 1 \pmod 2$, which is easily seen to be irreducible over $\mathbb{F}_2$: $f(x)$ has no roots in $\mathbb{F}_2$, and if it factors as the product of a quadratic and a cubic in $\mathbb{F}_2[x]$, then for some $a, b, c \in \mathbb{F}_2$:

$$f(x) = (x^2+ax+1)(x^3+bx^2+cx+1) \\ = x^5 + bx^4 + (ab+c)x^3 + (b+ac)x^2 + (a+c)x + 1,$$

which implies $b = c = 0$. But this gives that $f(x)$ has no quadratic term, a contradiction. Thus, $f(x)$ is irreducible over $\mathbb{Q}$. Using a computer algebra system, we determine that: $\mathrm{Disc}(f) = 3499417472929 = (7^2 38177)^2 \in (\mathbb{Q}^*)^2$. Thus, $\mathrm{Gal}(f/\mathbb{Q})$ is isomorphic to one of $A_5, D_5, \mathbb{Z}_5$. Using the Julia software system [Bez+14; Bez+12] with the `Roots` package implementing an algorithm by Zeng [Zen04] to numerically determine the roots of $f(x)$, we determine that

$$P_{10}(x) = x^{10} + 8x^9 - 210x^8 - 1531x^7 + 13948x^6 + 91971x^5 \\ - 268661x^4 - 1667396x^3 - 1284161x^2 - 1003871 + 64315,$$

which can readily be seen to be irreducible over $\mathbb{Q}$ using a computer algebra system. Thus, by Theorem 2, $\mathrm{Gal}(f/\mathbb{Q}) \cong A_5$. This contradicts the fact that $f(x)$ is an instance of a generic polynomial for $D_5$, and illustrates that even with 64 bit floating point numbers, it is impossible to use numerical methods to compute the roots of $f(x)$ in order determine $P_{10}(x)$ using (1). The reason is that, since $f(x)$ is irreducible over $\mathbb{Q}$, its roots are irrational, and it is only possible to finitely approximate irrational numbers numbers using floating point numbers. It follows that since the $\alpha_i + \alpha_j$ of (1) are irrational, they too can only finitely be approximated using floating point numbers, and so the polynomial obtained by implementing the definition of $P_{10}(x)$ given by (1) using floating-point arithmetic is at best an approximation of $P_{10}(x)$. Not only can irrationals only be finitely approximated, it is also of note is that many rationals can only be approximated; for example, the decimal number 0.1 has an infinite repeating binary representation and lies strictly between two floating-point numbers in base 2, neither of which exactly represent it [Gol91, p. 7]. At a more fundamental level, it is interesting to recall that, although Turing showed in 1937 [Tur37] that all algebraic numbers are computable (i.e., there exists a finite procedure to compute their decimal expansion), he also showed in the same paper that only a countable number of reals are computable, thereby limiting what numbers can be completely represented on a computer.

In short, it is clear that, in the context of Theorem 2, the decic $P_{10}(x)$ computed using (1) from roots of $f(x)$ determined by numerical methods cannot be used in place of the actual decic $P_{10}(x)$ of $f(x)$. The reader is referred to a paper by Goldberg [Gol91] for a deeper discussion of floating point arithmetic and its limitations.

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