EXPLORATIONS ON THE DIMENSION OF A GRAPH

RYAN KAVANAGH

ABSTRACT. Erdős, Harary and Tutte first defined the dimension of a graph G as the minimum number n such that G can be embedded into the Euclidean n-space E_n with every edge of G having length 1, see [3]. Although some have since extended the definition such that only adjacent vertices may be separated by a distance of 1[1], this paper will focus on the original definition. No general method is known for determining the dimension of an arbitrary graph G but lower and upper bounds will be proven for arbitrary graphs. A sharp upper bound will be given for k-partite graphs by generalising the proofs presented by Erdős et al. for bipartite graphs[3] and by Buckley et al. for tripartite graphs[1]. New findings further include a lower bound and applications to various classes of graphs.

1. INTRODUCTION

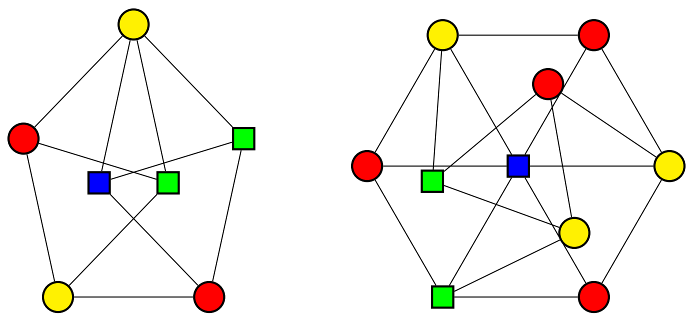
We define the dimension of a graph G, denoted $\dim G$, as the least number n such that G can be embedded into $\mathbb{R}^n$ with every edge of G having length 1, and we call such an embedding a unit-embedding. That's to say, if $\dim G = n$, there exists an injective mapping $f : V(G) \to \mathbb{R}^n, f(v_k) = (x_{k_1}, \dots, x_{k_n})$ such that if $v_k v_j \in E(G)$,

$$(x_{k_1} - x_{j_1})^2 + \cdots + (x_{k_n} - x_{j_n})^2 = 1.$$

The dimension of a graph is related to many open problems in discrete geometry. For example, the *Hadwiger-Nelson problem*, which seeks to determine the least number of colours needed such that no two points at unit distance from each other in the plane are of the same colour (see [7]), can be reduced to finding the largest possible chromatic number of a finite graph of dimension 2. By showing that the Golomb graph and the Moser spindle (both graphs of dimension 2, see figure 1) have chromatic number 4, a lower bound was set on the chromatic number of the plane. Generalisations of the Hadwiger-Nelson problem to higher dimensions n reduce to finding the largest possible chromatic number of a graph of dimension n.

The intriguing geometry behind the dimension of a graph and behind unit-embeddings will be explored in section 2 with examples such as platonic polyhedrals and antiprisms. We will also see in section 2 that the equilateral dimension of Euclidean space (the maximum number of equidistant points possible in a given E_n) is related to the dimension of complete graphs.

Since no systematic method for determining the dimension of an arbitrary graph is known, bounding the dimension of such graphs is an area of graph theory ripe for discoveries. A new lower bound will be presented in section 3. Further, known upper bounds



(A) Moser Spindle (B) Golomb graph

FIGURE 1. Graphs with chromatic number 4 and dimension 2

for k-partite graphs will be presented in section 4, and for arbitrary graphs in sections 5 and 6.

2. EXAMPLES

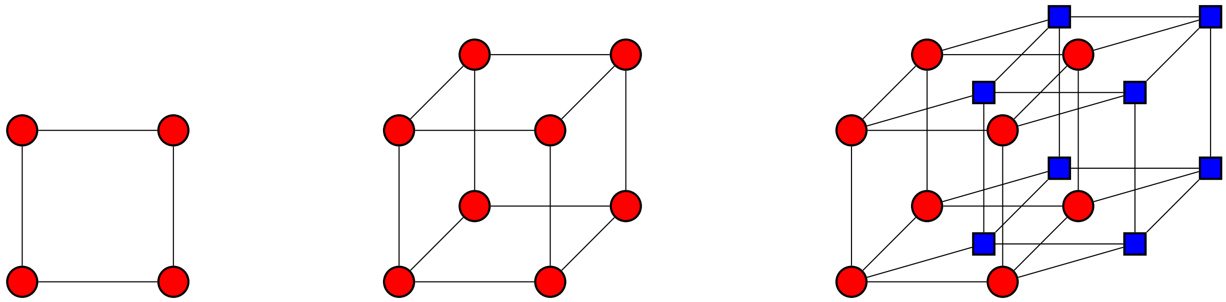
Let us first consider a few trivial examples. The dimension of any path is 1 while the dimension of any tree with three or more leaves is 2.

Example 1 (Platonic polyhedrals)**.** Define the platonic polyhedral graphs to be the the 3-regular graphs which are the skeletons of platonic solids. It is clear that the cube has dimension 2 (as evidence by figure 2b) and that the tetrahedron, octahedron, dodecahedron and icosahedron have dimension 3.

Example 2 (Prism graphs and Cartesian product with K_2)**.** Similarly to the cube, every prism graph (any 3-regular graph which is the skeleton of a prism in 3-space) has dimension 2. We notice that every prism graph has the same dimension as the identical cycles it has as bases. To generalise this finding to "prisms" with arbitrary graphs as bases, we must first introduce the Cartesian product of two graphs.

For two graphs G_1 and G_2, define the Cartesian product $G_1 \times G_2$ as the graph on the vertices $V(G_1) \times V(G_2)$. Two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Using the Cartesian product, we can formalise our definition of the cube as $C_4 \times K_2$ and the n-gonal prism graph as $C_n \times K_2$.

We can now generalise our findings for the prism graphs to any "prism" with a graph G as its bases by taking the Cartesian product of G and K_2. It is easy to see that $\dim G \times K_2 = \dim G$. As shown in figure 2, we can simply "make a copy" of G in $\mathbb{R}^{\dim G}$, that's to say, for every vertex v, create a vertex v' with the same coordinates and neighbours as v. Then translate every vertex in this copy by the same unit vector of $\mathbb{R}^{\dim G}$. After adding an edge from every vertex of G to its corresponding translated vertex, the resulting graph is clearly isomorphic to $G \times K_2$ and has the same dimension as G.



(A) C_4 (B) $C_4 \times K_2$ (C) $(C_4 \times K_2) \times K_2$

FIGURE 2. The Cartesian product with K_2 preserves dimension (all edges above have unit length)

Example 3 (Complete graphs)**.** Let us consider complete graphs. The complete graph on two vertices is a path and so $\dim K_2 = 1$. The complete graph on three vertices may be drawn as an equilateral triangle and so $\dim K_3 = 2$, while the complete graph on four vertices may be unit-embedded as the regular polyhedron in $\mathbb{R}^3$ and so $\dim K_4 = 3$. As we will prove in the following proposition, our intuition that, in general, $\dim K_n = n - 1$ is correct. The author is deeply grateful to Professor I. Dimitrov (Queen's University) for his assistance in proving that the maximum number of equidistant points in $\mathbb{R}^n$ is $n + 1$; any errors or inaccuracies in the following proof are the author's sole responsibility.

Proposition 1. *The dimension of the complete graph on n vertices, K_n, is $n - 1$.*

Proof. Since the vertex set of a unit-embedding of K_{k+1} is no more than a set of $k + 1$ equidistant points, it suffices to show that the least m such that $\mathbb{R}^m$ can contain such a set of points is $m = k$. Without loss of generality, we will assume that the distance between each pair of points is 1 and that we have one point, v_0, at the origin.

Let $\vec{v}_1, \ldots, \vec{v}_k$ denote the k vectors between the origin and the other k equidistant points, $v_1, \ldots, v_k$, in our set. By the hypothesis, $\vec{v}_i \cdot \vec{v}_i = 1$ for all $1 \leqslant i \leqslant k$. Further, since $\|\vec{v}_i - \vec{v}_j\| = 1$ for all $i \neq j$, we have

$$(\vec{v}_i \cdot \vec{v}_i)^2 - 2\,(\vec{v}_i \cdot \vec{v}_j) + (\vec{v}_j \cdot \vec{v}_j)^2 = 1$$

$$\vec{v}_i \cdot \vec{v}_j = \frac{1}{2}.$$

Using this, we can construct the Gram matrix $GM(\vec{v}_1, \ldots, \vec{v}_k)$, where

$$GM(\vec{v}_1, \ldots, \vec{v}_k) = \begin{vmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 & \ldots & \vec{v}_1 \cdot \vec{v}_k \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 & \ldots & \vec{v}_2 \cdot \vec{v}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_k \cdot \vec{v}_1 & \vec{v}_k \cdot \vec{v}_2 & \ldots & \vec{v}_k \cdot \vec{v}_k \end{vmatrix}$$

$$GM(\vec{v}_1, \ldots, \vec{v}_k) = \begin{vmatrix} 1 & \frac{1}{2} & \ldots & \frac{1}{2} \\ \frac{1}{2} & 1 & & \vdots \\ \vdots & & \ddots & \frac{1}{2} \\ \frac{1}{2} & \ldots & \frac{1}{2} & 1 \end{vmatrix}.$$

Results from linear algebra tell us that the determinant of $GM(\vec{v}_1, \ldots, \vec{v}_k)$ is non-zero if and only if $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

The matrix $GM(\vec{v}_1, \ldots, \vec{v}_k)$ has eigenvalue $1/2$ with multiplicity $k-1$. Recalling that the sum of the eigenvalues is the trace of the matrix, we see that the remaining eigenvalue, λ_k, is also non-zero since

$$\begin{aligned} \mathrm{tr}(GM(\vec{v}_1, \ldots, \vec{v}_k)) &= \lambda_1 + \cdots + \lambda_k \\ k &= (k-1)\frac{1}{2} + \lambda_k \\ \lambda_k &= \frac{k+1}{2}. \end{aligned}$$

Since the above matrix does not have 0 as an eigenvalue, its determinant is non-zero. This implies that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Since $\mathbb{R}^m$ can contain k linearly independent vectors if and only if $m \geqslant k$, we see that $k+1$ equidistant points, $v_0, \ldots, v_k$ can be embedded in $\mathbb{R}^m$ if and only if $m \geqslant k$.

It immediately follows that $\dim K_{k+1} = k$ and so $\dim K_n = n - 1$. □

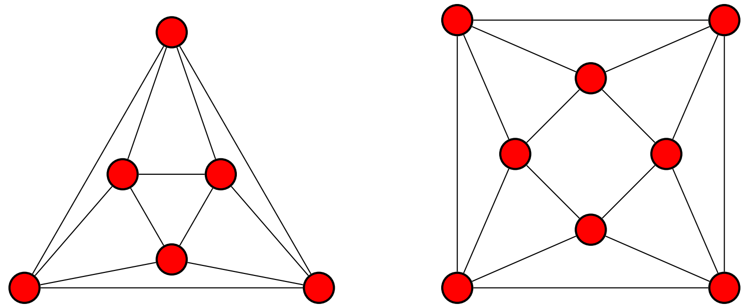
Example 4 (Antiprisms)**.** Finally, we'll define the n-gon antiprism graph on $2n$ vertices (the skeleton of the antiprism with the n sided polygon as its base) as the graph composed of two cycles C_n, an upper and a lower one, where there is an edge from each i-th vertex on the lower cycle to the i-th and the $(i+1 \mod n)$-th vertices on the upper cycle (see figure 3 for examples on six and eight vertices). It is then clear that each antiprism has dimension 3: the cycles can be embedded with unit length edges in parallel planes (they each have dimension 2) and the edges joining the cycles can be embedded with length 1, forming a series of equilateral triangles around the "circumference" of the antiprism.

3. Lower bounds

The following theorem and corollary establish a lower bound for the dimension of graphs.

Theorem 2. *For every graph* G, $\dim G \geqslant \dim H$ *for all subgraphs* H *of* G.

Proof. If G can be unit-embedded in $\mathbb{R}^{\dim G}$, then so can H. Thus, $\dim G \geqslant \dim H$. □



(A) 3-gon antiprism (B) 4-gon antiprism

FIGURE 3. Examples of antiprism graphs

Corollary 3. *For every graph* G*,* $\dim G \geqslant \omega(G) - 1$*.*

Proof. The clique number $\omega(G)$ is the order of the greatest clique of G. As seen in example 3, the dimension of the complete graph on n vertices, K_n, is $n-1$. Since $K_{\omega(G)} \subseteq G$, $\dim G \geqslant \omega(G) - 1$. □

4. Upper bound for k-partite graphs

We will proceed to generalise a theorem of Lenz[2] which Erdős, Harary and Tutte first presented for bipartite graphs and which was expanded to tripartite graphs by Buckley and Harary.

Theorem 4. *Any* k*-partite graph has dimension at most* $2k$*.*

Proof. By the definition of the dimension of a graph, it is sufficient to show that a unit-embedding of a k-partite graph in $\mathbb{R}^{2k}$ exists.

For each partition $1 \leqslant i \leqslant k$, assign to each of its vertices the distinct coordinates $(x_1, \ldots, x_{2k}) \in \mathbb{R}^{2k}$ such that $x_{2i-1}^2 + x_{2i}^2 = 1/2$ and all other x_j are zero.

It is then clear that the distance from any vertex in a partition i to any vertex in a partition j $(i \neq j)$ is $\sqrt{1/2 + 1/2} = 1$. □

5. Link to a graph's chromatic number

Theorem 4 leads to the following important link between a graph's dimension and its chromatic number:

Corollary 5. *For any graph* G*,* $\dim G \leqslant 2\chi(G)$*.*

Proof. Partition G into the $\chi(G)$-partite graph where every partition is of the same colour. □

This corollary is consistent with examples we've already seen, such as the Moser Spindle and Golomb graph of figure 1: both of these have dimension 2 and chromatic number 4. Similarly complete graphs on n vertices have dimension $n-1$ and chromatic number n and it is clear that that $n - 1 \leqslant 2n$.

6. Upper Bounds

We will next consider the upper bound on the dimension of an arbitrary graph presented by Maehara and Rödl in *On the Dimension to Represent a Graph by a Unit Distance Graph*[5].

Theorem 6 (Maehara and Rödl, 1990). *Let* G *be a graph with maximum degree* $\leqslant n$. *Then there exists a set of* $|V(G)|$ *unit vectors* $\{\bar{v} : v \in V(G)\}$ *in* $\mathbb{R}^{2n}$ *such that (1) any* $n+1$ *or fewer vectors are linearly independent, and (2)* $\bar{u}$ *and* $\bar{v}$ *are orthogonal if and only if* u *and* v *are adjacent.*

Proof. We will proceed by induction on $|V(G)|$. When $|V(G)| = 2$, it is clear that the graph may be embedded into $\mathbb{R}^1$. Let Δ be the maximum degree of G and u_0 a vertex of degree Δ. Suppose that the theorem is true for $G - u_0$, and let $\{\bar{v} : v \in V(G-u_0)\}$ be a set of unit vectors satisfying (1) and (2). Denote by $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ the linear subspace spanned by $\bar{v}_1, \ldots, \bar{v}_k$ and by $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle^{\perp}$ its orthogonal complement (every vector in $\mathbb{R}^{2n}$ that is orthogonal to every vector in $\langle \bar{v}_1, \ldots, \bar{v}_k \rangle$). Let $u_1, \ldots, u_\Delta$ be the neighbours of u_0 in G.

We want to show that we can find a unit vector $\bar{u}_0$ orthogonal to all those in $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle$. First we show that (3) for any $w_1, \ldots, w_n \in V(G-u_0)$, $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle^{\perp} \not\subseteq \langle \bar{w}_1, \ldots, \bar{w}_n \rangle$ (keeping in mind that $n \geqslant \Delta$). Suppose, on the contrary, that $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle^{\perp} \subset \langle \bar{w}_1, \ldots, \bar{w}_n \rangle$. Then $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle \supset \langle \bar{w}_1, \ldots, \bar{w}_n \rangle^{\perp}$, and since the vector space spanned by $\langle \bar{w}_1, \ldots, \bar{w}_n \rangle^{\perp}$ has dimension $2n - n \geqslant \Delta$, we must have $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle = \langle \bar{w}_1, \ldots, \bar{w}_n \rangle^{\perp}$. This implies that u_1 is adjacent to $w_1, \ldots, w_n$ by (2). But since u_1 is also adjacent to u_0 in G, the degree of u_1 in G is greater than Δ, a contradiction. Thus (3) holds. Hence,

$$\dim \langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle^{\perp} \cap \langle \bar{w}_1, \ldots, \bar{w}_n \rangle < \dim \langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle^{\perp} = 2n - \Delta.$$

Since it is impossible to cover a linear subspace of dimension $2n - \Delta$ by a finite number of linear subspaces of dimension $< 2n - \Delta$, there exists a unit vector $\bar{u}_0$ in $\langle \bar{u}_1, \ldots, \bar{u}_\Delta \rangle^{\perp}$ which does not lie in any linear subspace of the form

$$\langle \bar{w}_1, \ldots, \bar{w}_n \rangle \quad \text{or} \quad \langle \bar{u}_1, \ldots, \bar{u}_\Delta, \bar{w} \rangle^{\perp}.$$

Then the set of unit vectors

$$\{\bar{v} : v \in V(G - u_0)\} \cup \{\bar{u}_0\}$$

satisfies the conditions (1) and (2) for the graph G. □

Corollary 7 (Maehara and Rödl, 1990). *If* G *has maximum degree* $\Delta(G)$, *then* $\dim(G) \leqslant 2\Delta(G)$.

Proof. Set $n = \Delta(G)$ in Theorem 6 and let $V = \{\bar{v} : v(G)\}$ be a set of unit vectors in $\mathbb{R}^{2\Delta(G)}$ satisfying conditions (1) and (2) for G. Then the unit distance graph on the "point" set $\{(1/2)^{(1/2)}\bar{v} : v \in V\}$ in $\mathbb{R}^{2\Delta(G)}$ is clearly isomorphic to G. □

When dealing with k-partite graphs, the upper bound presented in theorem 4 is significantly better than the one in function of the maximum degree. Consider for example the complete tripartite-graph on $3 * 10^{1000}$ vertices. This graph's maximum degree is $2 * 10^{1000}$, and so by corollary 6, would have a dimension of at most $4 * 10^{1000}$. However, theorem 4 tells us that it has a dimension of at most 6, an outstanding improvement. We thus see that the upper bound presented in corollary 7 is far from being sharp.

Despite the example just given and that for every graph G, $\chi(G) \leqslant \Delta(G) + 1$, the looser upper bound given by corollary 7 is not without use. Since determining the chromatic number of a graph is a NP-complete problem (see [6]), Cobham's thesis implies that $\chi(G)$ is infeasible to compute if P$\neq$NP (see [4]).

7. Open problems

Since so little is known about the dimension of a graph, we leave the reader with the following open problems:

Open Question 1 (Erdős, Harary and Tutte)**.** A graph G is of critical dimension n if $\dim G = n$ and for any proper subgraph H, $\dim H < n$. Characterise graphs of critical dimension n for at least $n \geqslant 3$.

Open Question 2 (Kavanagh)**.** Given the arbitrarily large discrepancy between the upper bound of theorem 6 and the one found for k-partite graphs in theorem 4, can we find a better upper bound for arbitrary graphs?

Open Question 3 (Kavanagh)**.** Does a general method for obtaining the dimension of a graph exist, short of iteratively trying to unit-embed a graph in every Euclidean space between the relevant bounds? Can a graph's dimension be determined from its adjacency matrix?

Open Question 4 (Kavanagh)**.** In section 2, we saw how the dimension of a graph G is affected by its Cartesian product with the graph K_2. How is the dimension of a graph affected by other operations, such as complementation?

References

[1] Fred Buckley and Frank Harary, *On the Euclidean dimension of a wheel*, Graphs Combin. **4** (1988), no. 1, 23–30. MR922157 (88m:05074)

[2] Paul Erdős, *On sets of distances of* n *points in Euclidean space*, Publications of the Mathematical Institute of the Hungarian Academy of Sciences **5** (1960), 165–169.

[3] Paul Erdős, Frank Harary, and William T. Tutte, *On the dimension of a graph*, Mathematika **12** (1965), 118–122. MR0188096 (32 #5537)

[4] Steven Homer and Alan L. Selman, *Complexity theory*, Encyclopedia of Computer Science and Technology **26** (1992June), 77–100.

[5] Hiroshi Maehara and Vojtěch Rödl, *On the dimension to represent a graph by a unit distance graph*, Graphs Combin. **6** (1990), no. 4, 365–367. MR1092585 (91m:05176)

[6] Eric W. Weisstein, *Chromatic number*, MathWorld—A Wolfram Web Resource (2011). http://mathworld.wolfram.com/ChromaticNumber.html.

[7] ———, *Hadwiger-nelson problem*, MathWorld—A Wolfram Web Resource (2011). http://mathworld.wolfram.com/Hadwiger-NelsonProblem.html.